Math. Ann. (2011) 350:707–719
DOI 10.1007/s00208-010-0567-6

Mathematische Annalen

# Global well-posedness of the Cauchy problem for a super-critical nonlinear wave equation in two space dimensions

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Received: 2 December 2009 / Revised: 25 May 2010 / Published online: 15 September 2010


**Abstract** Extending the work of Ibrahim et al. (Commun Pure Appl Math 59(11): 1639–1658, 2006) on the Cauchy problem for wave equations with exponential nonlinearities in two space dimensions, we establish global well-posedness also in the super-critical regime of large energies for smooth, radially symmetric data.

## 1 Introduction

In [2], Ibrahim, Majdoub, and Masmoudi demonstrated that the initial value problem for the equation

$$u_{tt} - \Delta u + ue^{u^2} = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2 \tag{1}$$

is well-posed for smooth Cauchy data

$$(u, u_t)_{|t=0} = (u_0, u_1) \tag{2}$$

with initial energy

$$E(u(0)) = \int_{\mathbb{R}^2} e(u(0))dx \leq 2\pi, \tag{3}$$

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where

$$e(u) = \frac{1}{2}\left(|u_t|^2 + |\nabla u|^2 + e^{u^2} - 1\right). \tag{4}$$

Equation (1) is related to the critical Sobolev embedding in 2 space dimensions. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. Recall the Moser-Trudinger inequality

$$\sup_{u \in H_0^1(\Omega); ||\nabla u||^2_{L^2(\Omega)} \leq 1} \int_\Omega e^{4\pi u^2}\,dx < \infty; \tag{5}$$

see [6,11]. The exponent $\alpha = 4\pi$ is critical for this Orlicz space embedding in the sense that for any $\alpha > 4\pi$ there holds

$$\sup_{u \in H_0^1(\Omega); ||\nabla u||^2_{L^2(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2}\,dx = \infty. \tag{6}$$

On account of the obvious scaling property

$$\sup_{u \in H_0^1(\Omega); ||\nabla u||^2_{L^2(\Omega)} = 1} \int_\Omega e^{\alpha u^2}\,dx = \sup_{u \in H_0^1(\Omega); ||\nabla u||^2_{L^2(\Omega)} = \alpha} \int_\Omega e^{u^2}\,dx \tag{7}$$

and in view of (5), (6) the Cauchy problem for (1) with initial energy $E(u(0)) < 2\pi$ then may be regarded as *"sub-critical"*, while the cases $E(u(0)) = 2\pi$ or $E(u(0)) > 2\pi$ may be termed *"critical"* or *"super-critical"*, respectively.

The work [2] of Ibrahim, Majdoub, and Masmoudi thus shows that the Cauchy problem for Eq. (1) is well-posed in the subcritical and critical regimes, in agreement with the known results for nonlinear wave equations

$$u_{tt} - \Delta u + u|u|^{p-2} = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^n \tag{8}$$

in $n \geq 3$ space dimensions. For the latter problem well-posedness was shown to hold for $p \leq \frac{2n}{n-2}$, the critical exponent for the Sobolev embedding $H^1 \hookrightarrow L^p$ in $\mathbb{R}^n$, but little is known in the *"super-critical"* case when $p > \frac{2n}{n-2}$; see for instance the Notes in [7] for a brief survey and references.

It therefore may seem quite surprising that in the case of Eq. (1) the restriction (3) on the size of the initial data is unnecessary, at least in the radially symmetric case.

**Theorem 1.1** *For any radially symmetric data* $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \in C^\infty(\mathbb{R}^2)$ *there exists a unique, smooth solution* $u = u(t, |x|)$ *to the Cauchy problem* (1), (2)*, defined for all time.*

The proof of Theorem 1.1 is given in the next section. The final Sect. 3 contains a brief discussion of the result also in the context of nonlinear wave equations (8) in dimensions $n \geq 3$.

## 2 Proof of Theorem 1.1

Arguing indirectly, we suppose that the local solution $u$ to (1), (2) for certain Cauchy data $(u_0, u_1) = (u_0(|x|), u_1(|x|)) \in C^\infty(\mathbb{R}^2)$ cannot be smoothly extended to a neighborhood of some point $(T_0, x_0)$ where $T_0 \geq 0$.

Note that we may assume that $(u_0, u_1)$ are compactly supported, $T_0 > 0$, and that $u \in C^\infty([0, T_0[\times\mathbb{R}^2)$. Indeed, choose a function $\tau = \tau(|x|) \in C_0^\infty(\mathbb{R}^2)$ such that $\tau \equiv 1$ on a ball $B_R(0)$ for some $R > T_0 + |x_0|$. By [3] there exists a unique local solution $\tilde{u}$ to (1), (2) with Cauchy data $(\tau u_0, \tau u_1)$ and $\tilde{u} \in C^\infty([0, T_1[\times\mathbb{R}^2)$ for some maximal number $0 < T_1 \leq \infty$. This also follows by reasoning as in the proof of Lemma 2.1 below, using the fact that energy spreads with speed at most 1 in view of the energy inequality (10). Again on account of this fact the functions $u$ and $\tilde{u}$ then agree on the truncated light cone $\{(t, x);\ 0 \leq t < T_1, |x| + t < R\}$. It follows that $T_1 \leq T_0$; otherwise $\tilde{u}$ would yield a smooth extension of $u$ in a neighborhood of $(T_0, x_0)$.

Also observe that for any $C > 0$ radially symmetric functions $u \in C_0^\infty(\mathbb{R}^2)$ satisfying $||\nabla u||^2_{L^2} \leq C$ are locally uniformly bounded away from the origin. Thus, a first singularity can only occur at $x = 0$. Shifting time by $T_0$ and then reversing the arrow of time, in the following we may therefore assume that we have a compactly supported solution $u = u(t, |x|) \in C^\infty(]0, T_0] \times \mathbb{R}^2)$ of (1) blowing up at $(0, 0)$.

### 2.1 Energy inequality and flux decay

Upon multiplying (1) by $u_t$ we obtain the conservation law

$$0 = \frac{d}{dt}e(u) - div(\nabla u \cdot u_t) \tag{9}$$

for the energy density $e(u)$ and density of momentum

$$m(u) = \nabla u \cdot u_t.$$

Since clearly $|m(u)| \leq e(u)$, integration of (9) over a truncated light cone yields

$$E(u(t), B_R(x_0)) := \int_{B_R(x_0)} e(u(t))dx \leq E(u(s+t), B_{R+|s|}(x_0)) \tag{10}$$

for any $x_0 \in \mathbb{R}^2$, $R > 0$, and $0 < s + t, t \leq T_0$. In particular, energy will spread with speed at most 1. In fact, a sharper estimate holds. For ease of notation we state this estimate only in the case when $x_0 = 0$ which will be the only case of interest later. Denoting as $v(y) = u(|y|, y)$ the restriction of $u$ to the lateral boundary

$$M_S^T = \{z = (t, x); S \leq t \leq T, |x| = t\}$$

of the truncated forward light cone

$$K_S^T = \{z = (t, x);\, S \le t \le T, |x| \le t\}$$

with vertex at $z = (0, 0)$, and letting

$$Flux(u, M_S^T) := \frac{1}{2} \int\limits_{B_T \setminus B_S(0)} (|\nabla v|^2 + e^{v^2} - 1)dy,$$

upon integrating (9) over $K_S^T$ we find the identity

$$E(u(S), B_S(0)) + Flux(u, M_S^T) = E(u(T), B_T(0)) \tag{11}$$

for all $0 < S < T \le T_0$. In particular, $\lim_{T\downarrow 0} E(u(T), B_T(0))$ exists and we conclude decay of the flux

$$Flux(u, M_0^T) := \sup_{0<S<T} Flux(u, M_S^T) \to 0 \quad \text{as } T \downarrow 0. \tag{12}$$

Finally, from (10) we also have

$$E(u(T), B_T(0)) \le E(u(T_0), B_{T_0}(0)) \le E(u(T_0)) =: E_0 \tag{13}$$

for $0 < T < T_0$. We also denote $M^T = M_0^T$, $K^T = K_0^T$ for brevity.

### 2.2 Blow-up criterion

The work of Ibrahim, Majdoub, and Masmoudi [2] gives rise to the following characterization of blow-up through concentration of energy. In order to keep this paper self-contained, we include a short proof for completeness.

**Lemma 2.1** *There exists $\varepsilon_0 > 0$ such that*

$$E(u(T), B_T(0)) \ge \varepsilon_0 \quad \textit{for all } 0 < T \le T_0. \tag{14}$$

*Proof* Suppose $E(u(T), B_T(0)) < \pi/10$ for some $0 < T < T_0$. Then there is $R > 0$ such that $E(u(T), B_{T+R}(0)) \le \pi/10$, and by (10) there holds

$$E(u(t), B_{t+R}(0)) \le \pi/10 \quad \text{for all } 0 < t \le T. \tag{15}$$

As a consequence of (15) the functions $e^{u^2(t)}$ are uniformly bounded in $L^8(B_{t+R}(0))$ for $0 < t \le T$. To see this, first observe that for any $a > 0$, $b > 1$ we can estimate

$$ab \le (e^a - 1) + b \log b;$$

thus, for any $a, b > 0$ there holds

$$ab \leq (e^a - 1) + b \log(1 + b).$$

For any Lipschitz continuous cut-off function $0 \leq \tau \leq 1$ supported in $B_R(0)$ and any $0 < t \leq T$, upon letting $a = u^2(t)$, $b = |\nabla \tau|^2$ in the previous estimate we can therefore bound

$$\begin{aligned}\int_{B_R(0)} |\nabla(\tau u(t))|^2 dx &\leq 2 \int_{B_R(0)} (|\nabla u(t)|^2 + u^2(t)|\nabla \tau|^2)\, dx \\ &\leq 4E(u(t), B_R(0)) + 2 \int_{B_R(0)} |\nabla \tau|^2 \log(1 + |\nabla \tau|^2)\, dx.\end{aligned}$$

Choosing $\tau = \tau_L(|x|)$ with $\tau_L(r) = \min\{1, (\log\log\log\left(\frac{1}{r}\right) - \log\log\log L)_+\}$, for sufficiently large $L > 1/R$ we can achieve that

$$2 \int_{B_R(0)} |\nabla \tau|^2 \log(1 + |\nabla \tau|^2)\, dx \leq C \int_0^{1/L} \frac{\log(1/r)\, dr}{(\log\log(1/r)\log(1/r))^2\, r} \leq \frac{\pi}{10}.$$

For such $L$ then $\tau_L u(t) \in H_0^1(B_R(0))$ satisfies $||\nabla(\tau_L u(t))||^2_{L^2(B_R(0))} \leq \pi/2$ and the Moser-Trudinger inequality implies that $e^{\tau_L^2 u^2(t)}$ is bounded in $L^8(B_R(0))$, uniformly in $0 < t \leq T$. Since we can uniformly bound $|u(t, x)|$ at all points where $\tau_L(x) < 1$, it follows that the functions $e^{u^2(t)}$, indeed, are uniformly bounded in $L^8(B_{t+R}(0))$ for $0 < t \leq T$.

Let $Du_t = (u_{tt}, \nabla u_t)$ denote the space-time differential of $u_t$. Differentiating (1) in time and multiplying by $u_{tt}$, similar to (9) we obtain the identity

$$\frac{1}{2}\frac{d}{dt}|Du_t|^2 - div(\nabla u_t \cdot u_{tt}) + u_t(1 + 2u^2)e^{u^2} u_{tt} = 0. \tag{16}$$

By Young's inequality and the obvious estimate $1 + u^2 \leq e^{u^2}$ we can estimate

$$|u_t(1 + 2u^2)e^{u^2} u_{tt}| \leq Ce^{8u^2} + |u_t|^4 + |Du_t|^2,$$

where $C$ is an absolute constant. But an interpolation inequality of Gagliardo-Nirenberg-Ladyzhenskaya allows to bound

$$\int_{B_{t+R}(0)} |u_t|^4 dx \leq C \int_{B_{t+R}(0)} |u_t|^2 dx \cdot \int_{B_{t+R}(0)} (R^{-2}|u_t|^2 + |\nabla u_t|^2) dx. \tag{17}$$

Upon integrating (16) over the truncated cone $\{(t,x);\ 0 \le t < T, |x| < t+R\}$ and recalling that $e^{u^2(t)}$ is uniformly bounded in $L^8(B_{t+R}(0))$ for $0 < t \le T$, from Gronwall's inequality we now deduce the uniform bound

$$||Du_t(t)||_{L^2(B_{t+R}(0))} \le C = C(u(T)) \quad \text{for all } 0 < t \le T.$$

In view of (1), and again using that $e^{u^2(t)}$ is uniformly bounded in $L^8(B_{t+R}(0))$ for $0 < t \le T$, from standard elliptic estimates we then obtain also a uniform bound for $||\nabla^2 u(t)||_{L^2(B_{t+R/2}(0))}$, $0 < t \le T$, and therefore, by Sobolev's embedding, a uniform pointwise bound for $u$. Standard results now imply that $u$ may be smoothly extended to a neighborhood of $(0,0)$, contrary to hypothesis. □

The above argument also works in the non-symmetric case, where the threshold energy for blow-up might be of interest. Certainly our value $\varepsilon_0 = \pi/10$ may be improved. In fact, the results from [2,3] suggest that Lemma 2.1 holds true with $\varepsilon_0 = 2\pi$.

### 2.3 Pointwise estimates

In the radially symmetric setting with $v(y) = u(|y|, y) = v(|y|)$, for $0 < t < T_1 \le T_0$ by Hölder's inequality we can bound

$$\begin{aligned}|v(t)| &\le |v(T_1)| + \int_t^{T_1} |v'(s)|ds \le |v(T_1)| + \left(\int_t^{T_1} |\nabla v|^2 s ds \cdot \int_t^{T_1} \frac{ds}{s}\right)^{1/2} \\ &\le |v(T_1)| + Flux^{1/2}(u, M_t^{T_1}) \log^{1/2}(T_1/t).\end{aligned}$$

In view of (12) we may choose $0 < T_1 \le \min\{1, T_0\}$ such that $Flux^{1/2}(u, M_0^{T_1}) \le 1/3$ to ensure that for all $0 < t \le T_1$ there holds

$$Flux^{1/2}(u, M_t^{T_1}) \le Flux^{1/2}(u, M_0^{T_1}) \le 1/3.$$

We then fix $0 < T_2 \le T_1$ so that $6|v(T_1)| \le \log^{1/2}(1/t)$ for $0 < t \le T_2$. Also observing that $\log(T_1/t) \le \log(1/t)$ for our choice of $T_1$, we thus obtain the bound

$$|v(t)| \le \frac{1}{2} \log^{1/2}(1/t) \quad \text{for all } 0 < t \le T_2. \tag{18}$$

Again using Hölder's inequality, we now can extend the estimate (18) to the interior of the light cone. Indeed, for any $0 < \lambda \le 1$ and any $(t,x)$ with $0 < \lambda t \le |x| \le t \le T_2$

we can estimate

$$
\begin{aligned}
|u(t,|x|)| &\leq |u(t,t)| + \int_{|x|}^{t} |u_r(t,r)|dr \leq |v(t)| + \left( \int_{\lambda t}^{t} |\nabla u|^2 r dr \cdot \int_{\lambda t}^{t} \frac{dr}{r} \right)^{1/2} \\
&\leq \frac{1}{2} \log^{1/2}(1/t) + (E(u(t), B_t(0)))^{1/2} \log^{1/2}(1/\lambda) \\
&\leq \frac{1}{2} \log^{1/2}(1/t) + E_0^{1/2} \log^{1/2}(1/\lambda) \leq \log^{1/2}(1/t), \qquad (19)
\end{aligned}
$$

provided that $0 < t \leq T_3$ for suitable $0 < T_3 = T_3(\lambda) \leq T_2$.

Also observing that $t \log(1/t) \leq 1$ for all $t > 0$, we thus arrive at the following estimate.

**Lemma 2.2** *For any number $0 < \lambda \leq 1$ there exists a number $0 < T_3 = T_3(\lambda) \leq T_0$ such that for any $(t, x)$ with $0 < \lambda t \leq |x| \leq t \leq T_3$ there holds*

$$
|x|^2 u^2(t,x) e^{u^2(t,x)} \leq 1. \qquad (20)
$$

2.4 Exterior energy decay

By using a method of Shatah and Tahvildar-Zadeh [8], from Lemma 2.2 we deduce the following result.

**Lemma 2.3** *For any number $0 < \lambda \leq 1$ there holds*

$$
E(u(t), B_t(0) \backslash B_{\lambda t}(0)) \to 0 \quad \text{as } t \downarrow 0. \qquad (21)
$$

*Proof* For our radially symmetric solution $u$ of (1) we set

$$
e = \frac{1}{2}(u_r^2 + u_t^2 + e^{u^2}), \quad m = u_r \cdot u_t.
$$

Also letting

$$
F(u) = e^{u^2}, \; f(u) = 2ue^{u^2}, \quad L = \frac{1}{2}(u_r^2 - u_t^2 - F(u)) - rf(u)u_r,
$$

we compute

$$
\partial_t(re) - \partial_r(rm) = 0, \quad \partial_t(rm) - \partial_r(re) = L. \qquad (22)
$$

Changing coordinates to

$$
\xi = t + r, \quad \eta = t - r,
$$

and introducing the non-negative quantities $A$, $B$ with

$$A^2 = r(e - m), \quad B^2 = r(e + m),$$

the identities (22) turn into the equations

$$\partial_\xi A^2 = -L/2, \quad \partial_\eta B^2 = L/2, \tag{23}$$

respectively. Given any $0 < \lambda \leq 1$ now we observe that (20) for any $(t, r)$ with $0 < \lambda t \leq r \leq t \leq T_3$ permits to bound

$$r^2 f^2(u) = 4r^2 u^2 e^{u^2} F(u) \leq 4F(u).$$

Thus for such $(t, r)$ we have

$$\begin{aligned} L^2 &= \frac{1}{4}(u_r^2 - u_t^2 - F(u))^2 - (u_r^2 - u_t^2 - F(u)) r f(u) u_r + r^2 f^2(u) u_r^2 \\ &\leq \left(u_r^2 - u_t^2\right)^2 + F^2(u) + 2r^2 f^2(u) u_r^2 \leq \left(u_r^2 - u_t^2\right)^2 + F^2(u) + 8F(u) u_r^2 \\ &\leq \frac{C}{r^2} A^2 B^2 = C(e^2 - m^2) = C\left((u_r^2 - u_t^2)^2 + 2(u_r^2 + u_t^2)F(u) + F(u)^2\right). \end{aligned}$$

From (23) we then obtain the estimates

$$|\partial_\xi A| \leq \frac{C}{r} B, \quad |\partial_\eta B| \leq \frac{C}{r} A. \tag{24}$$

The conclusion now follows as in [8], Lemma 2.2. □

### 2.5 Time derivative decay

As in Corollary 2.3 of the work of Shatah and Tahvildar-Zadeh [8], Lemma 2.3 implies the decay of kinetic energy.

**Lemma 2.4** *We have*

$$\frac{1}{T} \int_{K^T} |u_t|^2 \, dz \to 0 \text{ as } T \downarrow 0. \tag{25}$$

*Proof* Multiplying (1) by $x \cdot \nabla u$ we obtain the identity

$$0 = \frac{d}{dt}(u_t \, x \cdot \nabla u) + div\left(\frac{x}{2}(|\nabla u|^2 - |u_t|^2 + e^{u^2}) - \nabla u \; x \cdot \nabla u\right) + |u_t|^2 - e^{u^2}.$$

Upon integrating this equation over $K_S^T$ and letting $S \to 0$, we find

$$\begin{aligned} \int_{K^T} |u_t|^2 \, dz &\le \int_{K^T} e^{u^2} \, dz - \int_{\{T\}\times B_T(0)} (u_t \; x \cdot \nabla u) dx + CT\, Flux(u, M_0^T) \\ &\le \int_{K^T} e^{u^2} \, dz + o(T), \end{aligned} \tag{26}$$

where $o(T)/T \to 0$ as $T \to 0$ on account of Lemma 2.3 and (12). Next we multiply (1) by $u/\log(1/t)$ to obtain the identity

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{u_t u}{\log(1/t)}\right) - div\left(\frac{u\nabla u}{\log(1/t)}\right) + \frac{|\nabla u|^2 - |u_t|^2}{\log(1/t)} \\ &\quad - \frac{u_t u}{t\log^2(1/t)} + \frac{u^2 e^{u^2}}{\log(1/t)}. \end{aligned} \tag{27}$$

Estimating $|u| \le |u - v| + |v|$, by Poincaré's inequality and (18) for $0 < t \le T \le T_2$ we can bound

$$\begin{aligned} \int_{\{t\}\times B_t(0)} |u_t u| dx &\le C\left(\int_{\{t\}\times B_t(0)} |u_t|^2 dx \cdot \int_{\{t\}\times B_t(0)} (|u-v|^2 + |v|^2) dx\right)^{1/2} \\ &\le Ct\left(\int_{\{t\}\times B_t(0)} |\nabla u|^2 dx + v^2(t)\right)^{1/2} \le CT\log^{1/2}(1/T); \end{aligned}$$

therefore

$$\int_{\{T\}\times B_T(0)} \frac{|u_t u|}{\log(1/T)} dx = o(T), \quad \int_{K^T} \frac{|u_t u|}{t\log^2(1/t)} dz \le C\frac{T}{\log^{1/2}(1/T)} = o(T).$$

Bounding the remaining terms in (27) in similar fashion, we finally obtain that

$$\int_{K^T} \frac{u^2 e^{u^2}}{\log(1/t)} dz \le o(T).$$

Together with (26) the latter estimate yields

$$\int_{K^T} |u_t|^2 dz \le \int_{K^T} \left(1 - \frac{u^2}{\log(1/t)}\right) e^{u^2} dz + o(T) \le o(T).$$

Here we observe that $\left(1-\frac{u^2}{\log(1/t)}\right) \leq 0$ unless $u^2 \leq \log(1/t)$; therefore

$$\int_{K^T} \left(1-\frac{u^2}{\log(1/t)}\right) e^{u^2} dz \leq \int_{K^T} \frac{1}{t} dz \leq CT^2.$$

□

2.6 Proof of Theorem 1.1

By Lemma 2.4 there is a sequence of numbers $T_k \downarrow 0$ as $k \to \infty$ such that for $t = T_k$ and $t = T_k/2$ there holds

$$\int_{\{t\}\times B_t(0)} |u_t|^2 dx \to 0 \quad \text{as } k \to \infty.$$

For such $T = T_k$ let

$$\bar{v} = \bar{v}_k = \fint_{B_T \setminus B_{T/2}(0)} v\, dy = \frac{4}{3\pi T^2} \int_{B_T \setminus B_{T/2}(0)} v\, dy.$$

Note that (18) implies that $|\bar{v}| \leq \log^{1/2}(1/T)$ for large $k$. Multiply (1) by $(u - \bar{v})$ to obtain the identity

$$0 = \frac{d}{dt}\left(u_t(u-\bar{v})\right) - div\left(\nabla u(u-\bar{v})\right) + |\nabla u|^2 - |u_t|^2 + u(u-\bar{v})e^{u^2}. \quad (28)$$

At $t = T$ and $t = T/2$ we estimate $|u - \bar{v}| \leq |u - v(t)| + |v(t) - \bar{v}|$ and observe that by Poincaré's inequality we can bound

$$\int_{\{t\}\times B_t(0)} |u - v(t)|^2\, dx \leq CT^2 \int_{\{t\}\times B_t(0)} |\nabla u|^2 dx$$

as well as

$$|v(t) - \bar{v}|^2 \leq \fint_{B_T \setminus B_{T/2}(0)} |v(t) - v|^2 dy \leq C\,Flux(u, M^T_{T/2}).$$

Thus, at $t = T$ and $t = T/2$ we have

$$\begin{aligned}
&\int\limits_{\{t\}\times B_t(0)} |u_t||u-\bar{v}|\,dx \\
&\le C\left(\int\limits_{\{t\}\times B_t(0)} |u_t|^2 dx \int\limits_{\{t\}\times B_t(0)} |u-\bar{v}|^2\,dx\right)^{1/2} \\
&\le CT\left(\int\limits_{\{t\}\times B_t(0)} |u_t|^2 dx\right)^{1/2}\left(\int\limits_{\{t\}\times B_t(0)} |\nabla u|^2 dx + Flux(u, M^T_{T/2})\right)^{1/2} = o(T).
\end{aligned}$$

Similarly, we find that

$$\int\limits_{B_T\setminus B_{T/2}(0)} |\nabla v||v-\bar{v}|\,dy \le CT\,Flux(u, M^T_{T/2}) = o(T).$$

Integrating (28) over $K^T_{T/2}$ we then obtain

$$\int\limits_{K^T_{T/2}} (|\nabla u|^2 - |u_t|^2 + u(u-\bar{v})e^{u^2})\,dz = o(T).$$

Finally, recalling (14) and observing that the bound $u(u-\bar{v}) \ge -|\bar{v}|^2/4 \ge -|\bar{v}|^2$ may be improved to yield $u(u-\bar{v}) \ge 1$ for $|u(z)| \ge 1 + |\bar{v}|$, we find

$$\begin{aligned}
T\varepsilon_0 &\le \int\limits_{K^T_{T/2}} (|\nabla u|^2 + |u_t|^2 + e^{u^2})\,dz \\
&\le \int\limits_{K^T_{T/2}} (|\nabla u|^2 - |u_t|^2 + u(u-\bar{v})e^{u^2})\,dz \\
&\quad + 2\int\limits_{K^T_{T/2}} |u_t|^2\,dz + \int\limits_{\{z\in K^T_{T/2};\,|u(z)|<1+|\bar{v}|\}} (1+|\bar{v}|^2)e^{(1+|\bar{v}|)^2} dz \\
&\le \int\limits_{K^T_{T/2}} (1+\log(1/T))e^{(1+\log^{1/2}(1/T))^2} dz + o(T) \le o(T).
\end{aligned}$$

For large $k$ a contradiction results, which proves the theorem.

## 3 Higher dimensions

We can also apply the preceding method to obtain global regularity of radially symmetric solutions to nonlinear wave equations (8) in $n \geq 3$ space dimensions. For example, when $n = 3$ a solution $u = u(t, r)$ to (8) induces a solution $v(t, r) = \sqrt{r} \cdot u(t, r)$ to the equation

$$v_{tt} - \Delta v + v \left( |u|^{p-2} + \frac{1}{4r^2} \right) = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2. \tag{29}$$

Letting

$$F = |v|^2 \left( |u|^{p-2} + \frac{1}{4r^2} \right), \quad f = 2v \left( |u|^{p-2} + \frac{1}{4r^2} \right),$$

then the inequality $r^2 f^2 \leq CF$ that was crucial for the proof of Lemma 2.3 holds if and only if with a uniform constant $C$ we can bound

$$|x|^2 |u(t, x)|^{p-2} \leq C \quad \text{for } 0 < |x| \leq 1. \tag{30}$$

By the Sobolev estimate

$$|u(|x|)| \leq |u(1)| + \int_{|x|}^{1} |u_r| dr \leq |u(1)| + \left( \int_{|x|}^{1} |\nabla u|^2 r^2 dr \cdot \int_{|x|}^{1} \frac{dr}{r^2} \right)^{1/2} \leq C|x|^{-1/2}$$

for a radially symmetric function $u = u(r) \in H^1(\mathbb{R}^3)$, clearly (30) is satisfied for any $p \leq 6$; in particular, the above method yields an alternative proof of the global regularity of radially symmetric solutions to the critical nonlinear wave equation

$$u_{tt} - \Delta u + u^5 = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^3, \tag{31}$$

established in [9]. (See the Notes in [7] for further references.) However, it is not clear how (30) can be shown to hold for any number $p > 6$, and this is only *one* of the issues that would have to be addressed in this case. With some luck, it might be possible to deal with logarithmically super-critical problems as treated in recent work of Tao [10].

The decisive difference between the 2-dimensional and the higher-dimensional cases seems to be that in 2 dimensions the super-critical regime is defined in terms of energy and not algebraically. Therefore the decay of the flux on the boundary of any light cone already puts us in a sub-critical regime on that part of the domain, and this is easily propagated to the interior of the cone by means of Sobolev's inequality as long as we keep a uniform distance (in the scale defined by the remainig time) from the axis $x = 0$.

Note however, that the recent results [1,3] of Ibrahim, Jrad, Majdoub, and Masmoudi show that for the local solution of the Cauchy problem (1), (2) there is no locally uniformly continuous dependence on the initial data in the energy norm when $E(u(0)) > 2\pi$, similar to nonlinear wave equations (8) with super-critical nonlinearities in dimensions $n \geq 3$. It therefore is not clear whether Theorem 1.1 may be extended also to non-symmetric data, even though it seems that a singularity would most likely appear in the radially symmetric case. Perhaps the recent work [4] of Ibrahim, Majdoub, Masmoudi, and Nakanishi on the scattering behavior of solutions to (1) with Cauchy data satisfying (3), or the references [1,3], and [5] can help provide further intuition for this problem and with regard to the issue of well-posedness and ill-posedness of super-critical wave equations in general.

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